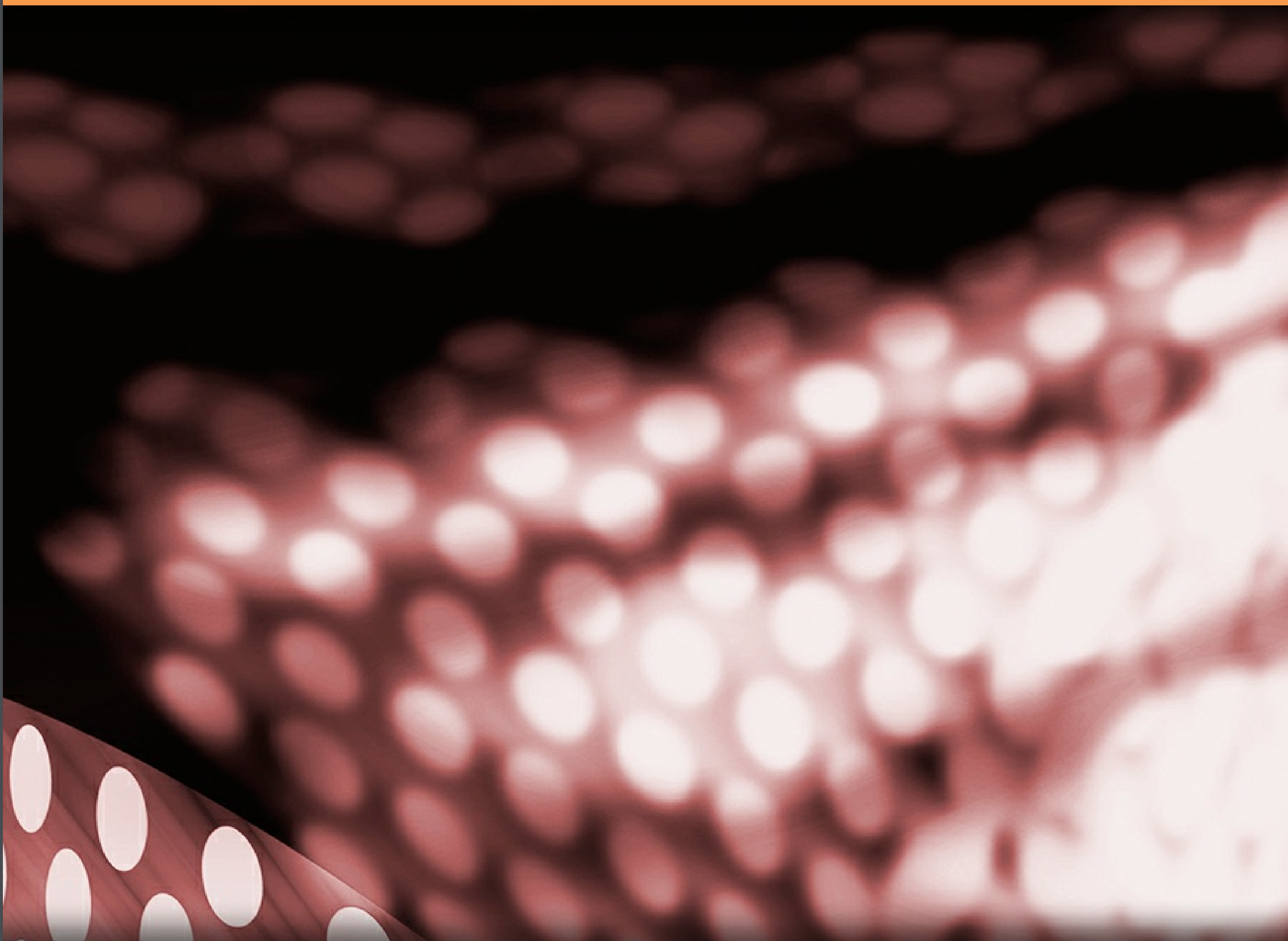
Introduction to Probability

Probability Examples c-1

Leif Mejlbro



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Probability Examples c-1 – Introduction to Probability

ISBN 978-87-7681-515-8

Contents

Introduction

This is the first book of examples from the *Theory of Probability*. This topic is not my favourite, however, thanks to my former colleague, Ole Jørsboe, I somehow managed to get an idea of what it is all about. The way I have treated the topic will often diverge from the more professional treatment. On the other hand, it will probably also be closer to the way of thinking which is more common among many readers, because I also had to start from scratch.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro
25th October 2009

1 Some theoretical background

It is not the purpose here to produce a full introduction into the theory, so we shall be content just to mention the most important concepts and theorems.

The topic probability is relying on the concept *σ-algebra*. A *σ-algebra* is defined as a collection $\mathcal{F}$ of subsets from a given set Ω, for which

1) The empty set belongs to the σ-algebra, $\emptyset \in \mathcal{F}$.

2) If a set $A \in \mathcal{F}$, then also its complementary set lies in $\mathcal{F}$, thus $\complement A \in \mathcal{F}$.

3) If the elements of a finite or countable sequence of subsets of Ω all lie in $\mathcal{F}$, i.e. $A_n \in \mathcal{F}$ for e.g. $n \in \mathbb{N}$, then the union of them will also belong to $\mathcal{F}$, i.e.

$$\bigcup_{n=1}^{+\infty} A_n \in \mathcal{F}.$$

The sets of $\mathcal{F}$ are called *events*.

We next introduce a *probability measure* on $(\Omega, \mathcal{F})$ as a set function $P : \mathcal{F} \to \mathbb{R}$, for which

1) Whenever $A \in \mathcal{F}$, then $0 \le P(A) \le 1$.

2) $P(\emptyset) = 0$ and $P(\Omega) = 1$.

3) If (A_n) is a finite or countable family of mutually disjoint events, e.g. $A_i \cap A_j = \emptyset$, if $i \neq j$, then

$$P\left(\bigcup_{n=1}^{+\infty} A_n\right) = \sum_{n=1}^{+\infty} P\left(A_n\right).$$

All these concepts are united in the *Probability field*, which is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a (non-empty) set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and P is a probability measure on $(\Omega, \mathcal{F})$.

We mention the following simple rules of calculations:

If $(\Omega, \mathcal{F}, P)$ is a probability field, and A, $B \in \mathcal{F}$, then

1) $\quad P(B) = P(A) + P(B\backslash) \ge P(A), \qquad$ if $A \subseteq B$.

2) $\quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

3) $\quad P(\complement A) = 1 - P(A)$.

4) $\quad$ If $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ and $A = \bigcup_{n=1}^{+\infty} A_n$, $\quad$ then $\quad P(A) = \lim_{n \to +\infty} P\left(A_n\right)$.

5) $\quad$ If $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ and $A = \bigcap_{n=1}^{+\infty} A_n$, $\quad$ then $\quad P(A) = \lim_{n \to +\infty} P\left(A_n\right)$.

Let $(\Omega, \mathcal{F}, P)$ be a probability field, and let A and $B \in \mathcal{F}$ be events where we assume that $P(B) > 0$. We define the *conditional probability of* A*, for given* B by

$$P(A \mid B) := \frac{P(A \cap B)}{P(B)}.$$

In this case, Q, given by

$$Q(A) := P(A \mid B), \qquad A \in \mathcal{F},$$

is also a probability measure on $(\Omega, \mathcal{F})$.

The *multiplication theorem of probability*,

$$P(A \cap B) = P(B) \cdot P(A \mid B).$$

Two events A and B are called *independent*, if $P(A \mid B) = P(A)$, i.e. if

$$P(A \cap B) = P(A) \cdot P(B).$$

We expand this by saying that n events A_j, $j = 1, \dots, n$, are independent, if we for any subset $J \subseteq \{1, \dots, n\}$ have that

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P\left(A_j\right).$$

We finally mention two results, which will become useful in the examples to come:

Given $(\Omega, \mathcal{F}, P)$ a probability field. We assume that we have a *splitting* $(A_j)_{j=1}^{+\infty}$ of Ω into events $A_j \in \mathcal{F}$, which means that the A_j are mutually disjoint and their union is all of Ω, thus

$$\bigcup_{j=1}^{+\infty} A_j = \Omega, \qquad \text{and} \qquad A_i \cap A_j = \emptyset, \quad \text{for every pair of indices } (i, j), \text{ where } i \neq j.$$

If $A \in \mathcal{F}$ is an event, for which $P(A) > 0$, then
The law of total probability,

$$P(A) = \sum_{j=1}^{+\infty} P\left(A_j\right) \cdot P\left(A \mid A_j\right),$$

and
Bayes's formula,

$$P\left(A_i \mid A\right) = \frac{P\left(A_i\right) \cdot P\left(A \mid A_i\right)}{\sum_{j=1}^{+\infty} P\left(A_j\right) \cdot P\left(A \mid A_j\right)}.$$

2 Set theory

Example 2.1 *Let A_1, A_2, ..., A_n be subsets of the sets Ω. Prove that*

$$\complement\left(\bigcup_{i=1}^{n} A_i\right) = \bigcap_{i=1}^{n} \complement A_i \qquad og \qquad \left(\bigcap_{i=1}^{n} \complement A_i\right) = \bigcup_{i=1}^{n} \complement A_i.$$

These formulæ are called de Morgan's formulæ.

1a. If $x \in \complement\left(\bigcup_{i=1}^{n} A_i\right)$, then x does not belong to any A_i, thus $x \in \complement A_i$ for every i, and therefore also in the intersection, so

$$\complement\left(\bigcup_{i=1}^{n} A_i\right) \subseteqq \bigcap_{i=1}^{n} \complement A_i.$$

1b. On the other hand, if $x \in \bigcap_{i=1}^{n} \complement A_i$, then x lies in all complements $\complement A_i$, so x does not belong to any A_i, and therefore not in the union either, so

$$\bigcap_{i=1}^{n} \complement A_i \subseteqq \complement\left(\bigcup_{i=1}^{n} A_i\right).$$

Summing up we conclude that we have equality.

2. If we put $B_i = \complement A_i$, then $\complement B_i = \complement\complement A_i = A_i$, and it follows from (1) that

$$\complement\left(\bigcup_{i=1}^{n} \complement B_i\right) = \bigcap_{i=1}^{n} B_i.$$

Then by taking the complements,

$$\bigcup_{i=1}^{n} \complement B_i = \complement\left(\bigcap_{i=1}^{n} B_i\right)$$

We see that (2) follows, when we replace B_i by A_i.

Example 2.2 *Let A and B be two subsets of the set Ω. We define the symmetric set difference $A\Delta B$ by*

$$A\Delta B = (A \setminus B) \cup (B \setminus A).$$

Prove that

$$A\Delta B = (A \cup B) \setminus (A \cap B).$$

Then let A, B and C be three subsets of the set Ω. Prove that

$$(A\Delta B)\Delta C = A\Delta(B\Delta C).$$

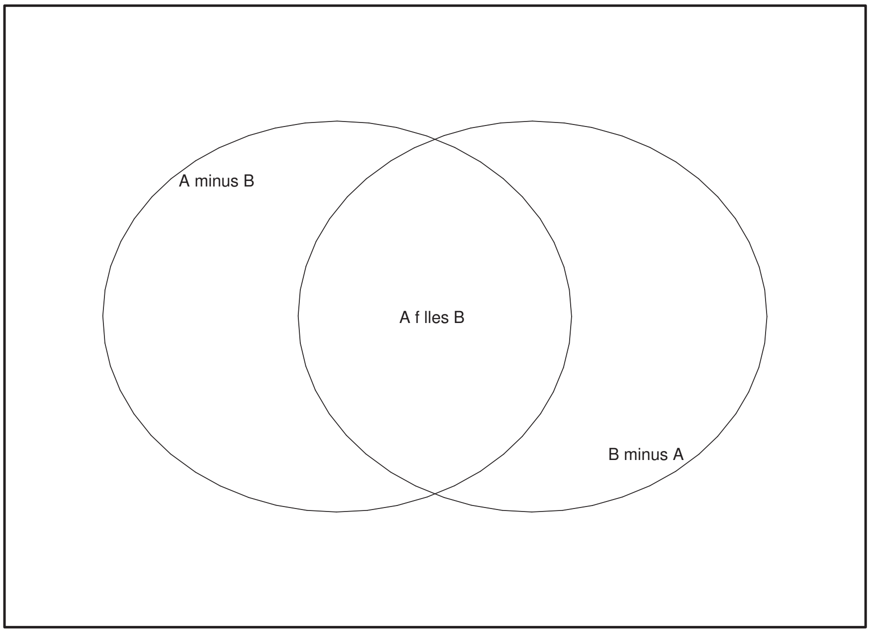


Figure 1: Venn diagram for two sets.

The claim is easiest to prove by a Venn diagram. Alternatively one may argue as follows:

1a. If $x \in (A \setminus B) \cup (B \setminus A)$, then x either lies in A, and not in B, or in B and not in A. This means that x lies in one of the sets A and B, but not in both of them, hence

$$A \Delta B = (A \setminus B) \cup (B \setminus A) \subseteqq (A \cup B) \setminus (A \cap B).$$

1b. Conversely, if $x \in (A \cup B) \setminus (A \cap B)$, and $A \neq B$, then x must lie in one of the sets, because $x \in A \cup B$ and not in both of them, since $x \notin A \cap B$, hence

$$(A \cup B) \setminus (A \cap B) \subseteqq (A \setminus B) \cup (B \setminus A) = A \Delta B.$$

1c. Finally, if $A = B$, then it is trivial that

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = \emptyset = (A \cup B) \setminus (A \cap B).$$

Summing up we get

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B).$$

2. If $x \in A \Delta B$, then x either lies in A or in B, and not in both of them. Then we have to check two possibilities:

(a) If $x \in (A \Delta B) \Delta C$ and $x \in (A \Delta B)$, then x does not belong to C, and precisely to one of the sets A and B, so we even have with equality that

$$\{(A \Delta B) \Delta C\} \cap (A \Delta B) = (A \setminus (B \cup V)) \cup (B \setminus (A \cup C)).$$

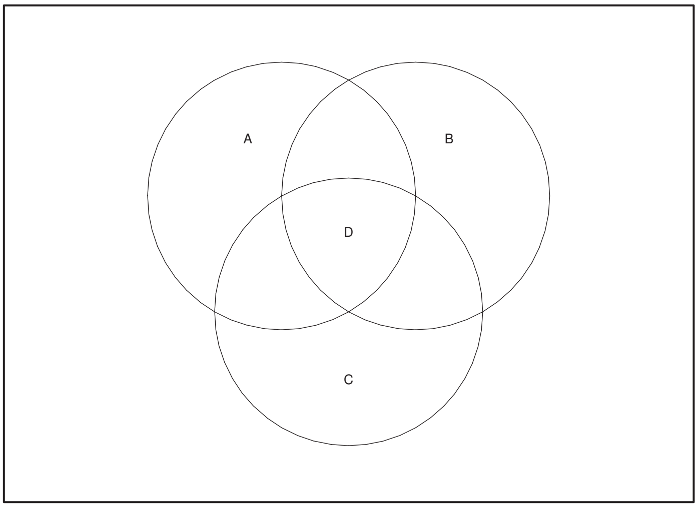


Figure 2: Venn diagram of three discs A, B, C. The set $(A\Delta B)\Delta C$ is the union of the domains in which we have put one of the letters A, B, C and D.

(b) If instead $x \in (A\Delta B)\Delta C$ and $x \in C$, then x does not belong to $A\Delta B$, so *either* x does not belong to any A, B, *or* x belongs to both sets, so we obtain with equality,

$$\{(A\Delta B)\Delta C\} \cap C = \{C \setminus (A \cup B)\} \cup \{A \cup B \cup C\}.$$

Summing up we get

$$\begin{array}{rcll} (A\Delta B)\Delta C & = & (A \setminus (B \cup C)) & \text{only contained in } A, \\ & & \cup (B \setminus (A \cup C)) & \text{only contained in } B, \\ & & \cup (C \setminus (A \cup B)) & \text{only contained in } C, \\ & & \cup (A \cap B \cap C) & \text{contained in all three sets.} \end{array}$$

By interchanging the letters we get the same right hand side for $A\Delta(B\Delta C)$, hence

$$(A\Delta B)\Delta C = A\Delta(B\Delta C).$$

3 Sampling with and without replacement

Example 3.1 *There are 10 different pairs of shoes in a wardrobe. Choose 4 shoes by chance. Find the probability of the event that there is at least one pair among them.*

First note that there are all together 20 shoes, from which we can choose 4 shoes in $\binom{20}{4}$ different ways.

We shall below give two correct and one false solution. The first (correct) solution is even given in two variants.

1) **Take the complements**, i.e. we apply that

$$P\{\text{at least one pair}\} = 1 - P\{\text{no pair}\}.$$

First variant.

First choice:	20 possibilities among 20 shoes:	$\frac{20}{20}$,	
Second choice:	18 possibilities of 19 shoes:	$\frac{18}{19}$,	(1 pair not allowed),
Third choice:	16 possibilities of 18 shoes:	$\frac{16}{18}$,	(2 pairs not allowed),
Fourth choice:	14 possibilities of 17 shoes:	$\frac{14}{17}$,	(3 pairs not allowed).

Summing up,

$$P\{\text{no pair}\} = \frac{20}{20}\cdot\frac{18}{19}\cdot\frac{16}{18}\cdot\frac{14}{17} = \frac{16\cdot 14}{19\cdot 17} = \frac{224}{323},$$

hence

$$P\{\text{at least one pair}\} = 1 - \frac{224}{323} = \frac{99}{323}.$$

Second variant. The four shoes stem from 4 pairs, where

- we can choose 4 pairs in $\binom{10}{4}$ ways.

Within each pair (*thus 4 times*) one can choose 1 shoe in $\binom{2}{1} = 2$ ways.

Then

$$P\{\text{no pair}\} = \frac{\binom{10}{4}\cdot 2^4}{\binom{20}{4}} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4}\cdot 2^4\cdot\frac{1\cdot 2\cdot 3\cdot 4}{20\cdot 19\cdot 18\cdot 17} = \frac{224}{323}.$$

Finally,

$$P\{\text{at least one pair}\} = 1 - \frac{224}{323} = \frac{99}{323}.$$

2) **Direct computation**.
The set of 4 shoes can contain 0, 1 or 2 pairs, hence

$$P\{\text{at least one pair}\} = P\{\text{precisely one pair}\} + P\{\text{precisely two pairs}\}.$$

We compute separately the two probabilities on the right hand side.

(a) $P\{\text{precisely one pair}\}$.

The pair can be chosen in $\binom{10}{1} = 10$ ways. This is done by drawing twice, so we still have to draw another two times.
Then, among the remaining pairs we can choose two in $\binom{9}{2}$ ways.
Within the latter two pairs we choose 1 shoe in $\binom{2}{1} = 2$ ways. Thus

$$P\{\text{precisely one pair}\} = \frac{10\cdot\binom{9}{2}\cdot 2^2}{\binom{20}{4}} = 10\cdot 4\cdot\frac{9\cdot 8}{1\cdot 2}\cdot\frac{1\cdot 2\cdot 3\cdot 4}{20\cdot 19\cdot 18\cdot 17} = \frac{96}{323}.$$

(b) $P\{\text{precisely two pairs}\}$.

Two pairs can be chosen in $\binom{10}{2}$ ways. Hence

$$P\{\text{precisely two pairs}\} = \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{10\cdot 9}{1\cdot 2}\cdot\frac{1\cdot 2\cdot 3\cdot 4}{20\cdot 19\cdot 18\cdot 17} = \frac{3}{323}.$$

Summing up it follows by an addition that

$$P\{\text{at least one pair}\} = \frac{96}{323} + \frac{3}{323} = \frac{99}{323}.$$

3) **Wrong argument**. The following is a frequently occurring wrong argument:

$$\text{“}P\{\text{at least one pair}\}\text{”} = \frac{10\binom{18}{2}}{\binom{20}{4}} = \frac{102}{323} \qquad \left(=\frac{6}{19}\right).$$

THE ERROR is that the possibility *“two pairs of shoes”* is counted twice by this procedure.

Example 3.2 *There are n different pairs of shoes in a wardrobe. Choose by chance $2r$ shoes, where $2r < n$. Find*

1) the probability of the event that there is no pair chosen,

2) the probability that there is precisely one pair among them.

We have in total $2n$ shoes, so we have $\binom{2n}{2r}$ possibilities.

If we introduce the notation

$$n^{(p)} = n\cdot(n-1)\cdots(n-p+1)$$

of p decreasing factors, then

$$\binom{2n}{2r} = \frac{(2n)^{(2r)}}{(2r)!}.$$

1) In this case the $2r$ shoes must come from $2r$ pairs. These $2r$ pair can be chosen in $\binom{n}{2r}$ ways.

Within each pair we can choose 1 shoe in $\binom{2}{1} = 2$ ways (in total $2r$). Hence

$$P\{\text{no pair}\} = \frac{\binom{n}{2r}\cdot 2^{2r}}{\binom{2n}{2r}} = \frac{n^{(2r)}\cdot 2^{2r}}{(2n)^{(2r)}}.$$

2) The pair can be chosen in $\binom{n}{1} = n$ ways, where we must use two draws.

We still have $n-1$ pairs left, of which we choose $2r-2$ in $\binom{n-1}{2r-2}$ ways.

We choose within each of the latter pairs 1 shoe in $\binom{2}{1} = 2$ ways, in total $2r-2$ ways. Hence

$$P\{\text{precisely one pair}\} = \frac{n\binom{n-1}{2r-2}\cdot 2^{2r-2}}{\binom{2n}{2r}} = \frac{n^{(2r-1)}\cdot 2^{2r-2}}{(2n)^{(2r)}}.$$

Example 3.3 *There are 12 parking places in a line on a parking space. To a given time 4 of the places are free. Find the probability that these 4 places are successive in the line.*
Answer the came question when the 12 parking spaces are lying in a circle.

The 4 places can be chosen in $\binom{12}{4} = 495$ ways.

We have 9 successes,

$$(1,2,3,4),\ (2,3,4,5),\ \dots,\ (9,10,11,12),$$

hence

$$P_1\{4 \text{ successive places}\} = \frac{9}{495} = \frac{1}{55}.$$

If the 12 parking spaces are placed in a circle, then we have the following 12 successes,

$$(1,2,3,4),\ \dots,\ (9,10,11,12),\ (10,11,12,1),\ (11,12,1,2),\ (12,1,2,3).$$

Consequently,

$$P_2\{4 \text{ successive places}\} = \frac{12}{495} = \frac{4}{165}.$$

Example 3.4 *We assume that the 5 digits of a car number are chosen randomly and independent of each other.*

1. *Find the probability that there are 5 different digits.*

2. *Find the probability that there are precisely 2 digits.*

3. *Find the probability that all 5 digits are equal.*

1a. *The same as (1) with the exception that the first digit must not be 0.*

If 0 is allowed as the first digit, we have all together 10^5 possibilities.

1. All digits can be different in $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways, hence

$$p_1 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{10^5} \approx 0.3024.$$

2. The positions of two equal digits can be chosen in $\binom{5}{2} = 10$ ways.
The common digit can be chosen in 10 ways.
The remaining three digits can be chosen in $9 \cdot 8 \cdot 7$ ways, hence

$$p_2 = \frac{10 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{10^5} \approx 0.5040.$$

3. There are only 10 ways, hence

$$p_3 = \frac{10}{10^5} = 0.0001.$$

1a. If the first digit cannot be 0, we get $9 \cdot 10^4$ possibilities. Furthermore, if we shall find the probability of that the 5 digits are different, then we have 9 possibilities for the first place, and $9 - 1 + 1 = 9$ possibilities for the second place (because we now can allow 0). For the remaining three places we get $8 \cdot 7 \cdot 6$, hence

$$q_1 = \frac{9 \cdot \cdot 9 \cdot 8 \cdot 7 \cdot 6}{9 \cdot 10^4} \approx 0.3024,$$

which is the same result as in (1).

Example 3.5 *Let $n > 3$. We randomly choose from the numbers $1, 2, 3, \dots, n$, in a sequence (without replacing the numbers), until they have all been taken.*

1) Find the probability that the numbers 1 and 2 are chosen successively in the given order.

2) Find the probability that the numbers 1, 2 and 3 are chosen successively in the given order.

We have several possibilities of solutions, of which we only give one.

First notice that the n numbers can be chosen in

$n!$ different orders.

Then assume that the numbers 1 and 2 are chosen successively (in the given order). In this way we "fix" two places, so we have in reality only $n - 1$ places to our disposition, hence we have

$(n - 1)!$ possibilities.

Analogously,

$(n - 2)!$ possibilities,

if we assume that 1, 2, 3 are chosen in the given order.

Summing up,

1) $P\{(1, 2) \text{ in the given order}\} = \dfrac{(n-1)!}{n!} = \dfrac{1}{n}.$

2) $P\{(1, 2, 3) \text{ in the given order}\} = \dfrac{(n-2)!}{n!} = \dfrac{1}{n(n-1)}.$

Example 3.6 *A man has n matches, which he breaks into a short and a long piece. He collects the $2n$ pieces at random in n pairs.*
Find the probability that each pair is consisting of a short and a long piece, and find in particular the probability in the case $n = 5$.

When he breaks all matches he gets in total $2n$ pieces, of which n are long and the remaining n are short. From these $2n$ pieces he successively picks them up one by one, giving $(2n)!$ possible ordered strings.
Let us now consider the successes.
The first piece can be chosen in $2n$ ways.
The next piece much be chosen from the n "complementary" pieces, giving n possibilities.
Since we have already chosen two pieces, the third piece can be chosen in $2n - 2$ ways.
The fourth piece must be chosen among the remaining $n - 1$ "complementary" pieces.
We continue in this way, so we end with

$$\{2n \cdot n\} \cdot \{(2n-2) \cdot (n-1)\} \cdots \{2 \cdot 1\} = 2^n \cdot (n!)^2$$

successes.
Summing up we get

$$P\{\text{each pair consiste of a short and a long piece}\} = p_n = \frac{2^n \cdot (n!)^2}{(2n)!} = \frac{2^n}{\begin{pmatrix} 2n \\ n \end{pmatrix}}.$$

In particular,

$$p_5 = \frac{2^5}{\begin{pmatrix} 10 \\ 5 \end{pmatrix}} = \frac{32 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = \frac{8}{63}.$$

Example 3.7 *A lift starts with 7 passengers and it stops at 10 storeys. Find the probability that the passengers get off the lift at 7 different storeys. We assume that each passenger has the probability $1/10$ to get off at any given storey and that the storeys at which each passenger are independent of each other.*

The total number of possibilities is 10^7.

The total number of successes is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$.

Hence

$$P\{\text{7 different storeys}\} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{10^7} = \begin{pmatrix} 10 \\ 7 \end{pmatrix} \cdot \frac{7!}{10^7} = 0.06048.$$

4 Playing cards

Example 4.1 *Playing bridge North has no ace. Find the probability that South has precisely 2 aces.*

North has 13 cards, none of then an ace.
We shall deal 4 aces and 35 other cards among the three remaining players.
Since South must have two aces and 11 other cards, we have

$$P\{\text{South has 2 aces}\} = \frac{\binom{4}{2}\binom{35}{11}}{\binom{39}{13}} = \frac{26 \cdot 25}{37 \cdot 19 \cdot 3} = 0.3082.$$

Example 4.2 *Playing bridge, one of the pairs of partners has in total 9 hearts. Find the probability that the remaining 4 hearts among the other pair of partners is distributed as follows*

1) 2 to each,

2) 1 to one of them and 3 to the other one,

3) 4 to the same player.

This example can be solved in many ways. We give three variants.

First variant. The two players have in total 26 cards, of which 4 are hearts.

1) $P\{2\text{–}2\}$:

   a) The 26 cards are distributed in two talons of 13 cards each in $\binom{26}{13}$ ways.

   b) The 22 cards which are not heart can be distributed in two talons of 11 in each of them in $\binom{22}{11}$ ways.

   c) The 4 hearts are distributed in two talons of 2 in each in $\binom{4}{2}$ ways.

   Summing up,

$$P\{2\text{–}2\} = \frac{\binom{4}{2}\binom{22}{11}}{\binom{26}{13}} = 6 \cdot \frac{22!}{11!11!} \cdot \frac{13!13!}{26!} = \frac{6 \cdot 13 \cdot 12 \cdot 13 \cdot 12}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{234}{575}.$$

2) $P\{3\text{–}1\}$:

   a) As above we get $\binom{26}{13}$ ways.

   b) The 22 cards which are not hearts are distributed in two talons of 10 in one of them and 12 in the other one in

$$\binom{22}{10} = \binom{22}{12} \text{ ways.}$$

   c) The 4 hearts are distributed in two talons of 3 in one of them and 1 in the other one in $\binom{4}{3} = \binom{4}{1} = 4$ ways.

   d) There are 2 partners, thus 3–1 is a short symbol for both 3–1 and 1–3. Hence, we have 2 possibilies.

   Summing up we get

$$P\{3\text{–}1\} = 2 \cdot \frac{\binom{4}{1}\binom{22}{10}}{\binom{26}{13}} = \frac{1 \cdot 4 \cdot 22!}{10!12!} \cdot \frac{13!13!}{26!} = \frac{2 \cdot 4 \cdot 13 \cdot 12 \cdot 11 \cdot 13}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{286}{575}.$$

3) $P\{4\text{–}0\}$. As in (2) we get analogously

a) $\binom{26}{13}$ ways,

b) $\binom{22}{9}$ ways,

c) $\binom{4}{0} = 1$ way,

d) $\binom{2}{1} = 2$ ways.

Summing up we get

$$P\{4\text{–}0\} = 2 \cdot \frac{\binom{4}{0}\binom{22}{9}}{\binom{26}{13}} = \frac{2 \cdot 22!}{9!13!} \cdot \frac{13!13!}{26!} = \frac{2 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{55}{575} = \frac{11}{115}.$$

Check:

$$P\{2\text{–}2\} + P\{3\text{–}1\} + P\{4\text{–}0\} = \frac{234}{575} + \frac{286}{575} + \frac{55}{575} = 1,$$

and the found probabilities have the sum 1. $\Diamond$

Second variant. The 4 hearts can be distributed on 26 places in $\binom{26}{4}$ ways.

1) $P\{2\text{–}2\}$.

The number of successes is $\binom{13}{2}\binom{13}{2}$, hence

$$P\{2\text{–}2\} = \frac{\binom{13}{2}\binom{13}{2}}{\binom{26}{4}} = \frac{13^2 \cdot 12^2}{1^2 \cdot 2^2} \cdot \frac{1 \cdot 2 \cdot 3 \cdot 4}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{13 \cdot 12 \cdot 3}{2 \cdot 25 \cdot 23} = \frac{234}{575}.$$

2) $P\{3\text{–}1\}$.

Since we have a pair of partners, there are $2\binom{13}{3}\binom{13}{1}$ successes, hence

$$P\{3\text{–}1\} = 2 \cdot \frac{\binom{13}{1}\binom{13}{3}}{\binom{26}{4}} = 2 \cdot \frac{13}{1} \cdot \frac{13 \cdot 12 \cdot 11}{1 \cdot 2 \cdot 3} \cdot \frac{1 \cdot 2 \cdot 3 \cdot 4}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{2 \cdot 13 \cdot 11}{25 \cdot 23} = \frac{286}{575}.$$

3) $P\{4\text{–}0\}$.

The number of successes is $2 \cdot \binom{13}{4}\binom{13}{0}$, hence

$$P\{4\text{–}0\} = 2 \cdot \frac{\binom{13}{0}\binom{13}{4}}{\binom{26}{4}} = \frac{2 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{1 \cdot 2 \cdot 3 \cdot 4}{26 \cdot 25 \cdot 24 \cdot 23} = \frac{55}{575} = \frac{11}{115}.$$

Third variant. We chase the distribution of the hearts at S = South and N = North:

1) $P\{2\text{–}2\}$.
   1. A heart is placed at **S** with probability $\dfrac{13}{26} = \dfrac{1}{2}$.
   2. A heart is placed at **S** with probability $\dfrac{12}{25}$.
   3. A heart is placed at **N** with the probability $\dfrac{13}{24}$.
   4. A heart is placed at **N** with probability $\dfrac{12}{23}$.

   S and **N** can be chosen in $\begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6$ ways, thus

   $$P\{2\text{–}2\} = 6 \cdot \frac{1}{2} \cdot \frac{12}{25} \cdot \frac{13}{24} \cdot \frac{12}{23} = \frac{234}{575}.$$

2) Analogously,

   $$P\{3\text{–}1\} = 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \frac{1}{2} \cdot \frac{12}{25} \cdot \frac{11}{24} \cdot \frac{13}{23} = \frac{286}{575}.$$

3) Finally, and also analogously,

   $$P\{4\text{–}0\} = 2 \begin{pmatrix} 4 \\ 4 \end{pmatrix} \cdot \frac{1}{2} \cdot \frac{12}{25} \cdot \frac{11}{24} \cdot \frac{10}{23} = \frac{55}{575} = \frac{11}{115}.$$

Example 4.3 *Assume that North and South together have 10 trumps. Find the probability that the remaining 3 trumps either all are at East or all are at West.*

East and West have 26 cards together, of which 3 are trumps.
When we consider the distribution we get

$$P\{\text{East has all three trumps}\} = \frac{1 \cdot \begin{pmatrix} 23 \\ 10 \end{pmatrix}}{\begin{pmatrix} 26 \\ 13 \end{pmatrix}} = \frac{11}{100}.$$

By symmetry, this is also the probability for West having all three trumps.
Hence,

$$P\{\text{all 3 trumps on the same hand}\} = 2 \cdot \frac{11}{100} = \frac{11}{50} = 22\%.$$

Example 4.4 *By a dealing the 52 cards are taken one by one. Find the probability that the four aces are succeeding each other.*

We can choose 4 positions out of 52 possibilities in $\binom{52}{4}$ ways.

We see that 4 in a row: $(1,2,3,4)$, $(2,3,4,5)$, $\dots$, $(49,50,51,52)$, can occur 49 times.

Then

$$P\{4 \text{ aces}\} = \frac{49}{\binom{52}{4}} = \frac{1}{5525} \approx 0.000\,181.$$

Example 4.5 *Find the probability that each of the 4 bridge players get precisely one ace.*
Find the probability in 7 games that at least one of these 7 games have this uniform distribution of the aces, and find also the probability that precisely one of the 7 games has this uniform distribution of the aces.

First note that 52 cards can be distributed between 4 players with 13 cards to each in

$$\frac{52!}{13!\,13!\,13!\,13!} \text{ ways.}$$

If we assume that each player has precisely one ace and 12 other cards, this can be done in

$$4! \cdot \frac{48!}{12!\,12!\,12!\,12!} \text{ ways.}$$

Then we derive the probability of each player having one ace,

$$p = \frac{4!48!(13!)^4}{(12!)^4 52!} = \frac{24 \cdot 13^4}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{13^3}{49 \cdot 25 \cdot 17} = 0.1055.$$

Then

$$P\{\text{none of the 7 games has this uniform distribution of the aces}\} = (1-p)^7 = 0.4582,$$

hence

$$P\{\text{at least one of the 7 games has the uniform distribution}\} = 1-(1-p)^7 = 0.5418.$$

Finally,

$$P\{\text{precisely 1 of the 7 games has this uniform distribution}\} = 7 \cdot p(1-p)^6 = 0.3783.$$

Alternatively one may consider 52 places divided into 4 blocks of 13 in each of them.
The 4 aces should be placed in given 4 of the 52 places, which gives the total number of possibilities,

$$m = \binom{52}{4} = 270\,725.$$

Then consider the distribution of the aces between the blocks.

The distribution 1-1-1-1.
We choose in each block precisely 1 of the 13 places, so the number of possibilities is

$$g_1 = \binom{13}{1}^4 = 13^4,$$

hence,

$$p_1 = P\{1-1-1-1\} = \frac{g_1}{m} = \frac{13^4 \cdot 4!}{52 \cdot 51 \cdot 50 \cdot 49} = 0.1055.$$

The distribution 2-1-1-0.
The block that should contain 2 aces is chosen among the 4 blocks. It remains 3 possibilities for choosing the block, which does not contain any ace. This gives the following number of possibilities,

$$g_2 = 4 \cdot 3 \cdot \binom{13}{2}\binom{13}{1}^2\binom{13}{0} = 72 \cdot 13^3 = 158\,184,$$

hence

$$p_2 = P\{2-1-1-0\} = \frac{g_2}{m} = 0.5843.$$

The distribution 2-2-0-0.
The number of successes is

$$g_3 = \binom{13}{2}^2 \cdot \binom{13}{0}^2 \cdot \binom{4}{2} = 36\,504,$$

thus

$$p_3 = P\{2-2-0-0\} = \frac{g_3}{m} = 0.1348.$$

The distribution 3-1-0-0.
The number of successes is

$$g_4 = 4 \cdot 3 \cdot \binom{13}{3} \cdot \binom{13}{1} \cdot \binom{13}{0} = 44\,616,$$

thus

$$p_4 = P\{3-1-0-0\} = \frac{g_4}{m} = 0.1648.$$

The distribution 4-0-0-0.
The number of successes is

$$g_5 = 4 \cdot \binom{13}{4} \cdot \binom{13}{0}^4 = 2\,860,$$

hence the probability becomes

$$p_5 = P\{4-0-0-0\} = \frac{g_5}{m} = 0.0106.$$

Control. It follows that

$$p_1 + p_2 + p_3 + p_4 + p_5 = 0.1055 + 0.5843 + 0.1348 + 0.1648 + 0.0106 = 1.0000. \qquad \Diamond$$

Example 4.6 *Find the probability that North in a game of bridge gets 4 aces. Find the smallest number of games n, for which the probability of North having 4 aces in at least one of the n games is bigger than $\frac{1}{2}$.*

North can in total obtain $\binom{52}{13}$ different hands.

The successes are characterized by 4 aces and 9 other cards, so there are $\binom{4}{4} \cdot \binom{48}{9}$ successes.
Thus, the probability for North obtaining 4 aces in one game is

$$p = \frac{\binom{48}{9}}{\binom{52}{13}} = \frac{48!\,13!\,39!}{52!\,39!\,9!} = \frac{13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{11}{49 \cdot 17 \cdot 5} = \frac{11}{4165} = 0.02641.$$

Hence by the complementary,

$$P\{\text{N does } \textit{not} \text{ obtain 4 aces in a game}\} = 1 - \frac{11}{4165} = \frac{4154}{4165} = 0.997359,$$

so

$$P\{\text{N never obtains 4 aces in } n \text{ games}\} = 0.997359^n.$$

This probability is $< \frac{1}{2}$, when

$$n \cdot \ln 0 - 997359 < -\ln 2 \quad (< 0),$$

i.e. when

$$n > \frac{-\ln 2}{\ln 0 - 997359} = 267, 10.$$

We conclude that the smallest possible n is 263.

Remark 4.1 The control shows that

$$\left(\frac{4154}{4165}\right)^{263} = 0.4988,$$

and

$$\left(\frac{4154}{4165}\right)^{262} = 0.5001.$$

5 Miscellaneous

Example 5.1 *Find the probability of the event that the birthdays of 6 randomly chosen persons are distributed in precisely 2 months (i.e. such that precisely 10 of the months do not contain any birthday.) We assume that every month can be chosen with the same probability.*

The 6 birthdays can be distributed in 12^6 ways on the 12 months.

The 2 months can be chosen in $\binom{12}{2} = 66$ ways.

The 6 birthdays can be distributed in $2^6 = 64$ ways inside the 2 months where, however, 2 ways are not allowed, because we do not permit that all anniversaries lie only lie in 1 month. Hence we obtain $64 - 2 = 62$ ways.

As a conclusion the probability becomes

$$\frac{66 \cdot 62}{12^6} = \frac{11 \cdot 31}{12^5} = \frac{341}{12^5} \approx 0.00137.$$

Alternatively we may apply the following tedious argument:

The probability of the first two persons having their birthdays in different months, while the remaining four persons have their birthdays in the thus determined months is

$$\frac{12}{12} \cdot \frac{11}{12} \cdot \left(\frac{2}{12}\right)^4 = \frac{11}{12} \cdot \frac{1}{6^4}.$$

The probability that the first two persons have their birthdays in the same month, the third person in a different month and the remaining three persons in the thus determined months is

$$\frac{12}{12} \cdot \frac{1}{12} \cdot 1112 \cdot \left(\frac{2}{12}\right)^3 .$$

The probability that the first three persons have their birthdays in the same month, the fourth person in a different month, while the remaining two persons have their birthdays in the thus determined months is

$$1 \cdot \left(\frac{1}{12}\right)^2 \cdot \frac{11}{12} \cdot \left(\frac{1}{12}\right)^2 .$$

The probability that the first four have their birthdays in the same month, the fifth person in a different month, while the remaining person has his birthday in one of the thus determined months is

$$1 \cdot \left(\frac{1}{12}\right)^3 \cdot \frac{11}{12} \cdot \frac{2}{12}.$$

The probability that the first five persons have their birthdays in the same month, while the remaining person has his birthday in any other month is

$$1 \cdot \left(\frac{1}{12}\right)^4 \cdot \frac{11}{12}.$$

The searched probability is obtained by addition,

$$\frac{11}{12} \cdot \left(\frac{2}{12}\right)^4 + \frac{1}{12} \cdot \frac{11}{12} \cdot \left(\frac{2}{12}\right)^3 + \left(\frac{1}{12}\right)^2 \cdot \frac{11}{12} \cdot \left(\frac{1}{12}\right)^2 + \left(\frac{1}{12}\right)^3 \cdot \frac{11}{12} \cdot \frac{2}{12} + \left(\frac{1}{12}\right)^4 \cdot \frac{11}{12} = \frac{341}{12^5}.$$

Example 5.2 de Méré's paradox.

1) *Four dices are thrown once. Find the probability that we obtain at least one six.*

2) *Now, perform 24 throws with 2 dices. Find the probability that* at least *one of the 24 throws results in two sixes.*

A French gambler, Chevalier de Méré, believed that these two probabilities should be equal to $\frac{2}{3}$*, so he lost a lot of money on betting on this.*

Whenever one of the phrases *"at least"* or *"at most"* occurs it is usually easier to consider the *complementary event.*

1) We get from

$$P_4\{\text{no sixes}\} = \left(\frac{5}{6}\right)^4,$$

that

$$P_4\{\text{at least one six}\} = 1 - P_4\{\text{no sixes}\} = 1 - \left(\frac{5}{6}\right)^4 = 1 - \frac{625}{1296} = \frac{671}{1296} \approx 0.518.$$

2) In the same way we get

$$P_{24}\{\text{no double sixes}\} = \left(\frac{35}{36}\right)^{24},$$

hence

$$P_{24}\{\text{at least one double sixes}\} = 1 - P_{24}\{\text{no double sixes}\} = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.491.$$

It follows immediately that the two probabilities are different.

Remark 5.1 The result can also be computed directly. We shall show this on (1), in which case

$$\begin{aligned}
&P_4\{\text{at least one six}\}\\
&\qquad = P_4\{\text{first six in throw number 1}\} + P_4\{\text{first six in throw number 2}\}\\
&\qquad\quad + P_4\{\text{first six in throw number 3}\} + P_4\{\text{first six in throw number 4}\}\\
&\qquad = \frac{1}{6} + \frac{5}{6}\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^3\cdot\frac{1}{6} = \frac{671}{1296}.
\end{aligned}$$

Example 5.3 *Spreading of rumours.*
In a town of $n+1$ inhabitants one person is telling a rumour to another one, who tells the story to another one etc.. At each step the rumour-monger randomly chooses the person who he or she is going to tell the rumour.

1) Find the probability of that the romour is told r times without returning to the person who started gossiping.

2) Find the probability that the rumour is told r times, where $r < n$, without returning to anyone who has already received the rumour.

It is here allowed that the rumour can also be told to the person who has just told it to himself, cf. Example 5.4.

1) The first person cannot tell the story to himself, so when $r = 1$, the probability is 1.

   The following persons have $n-1$ choices of n possibilities, thus the probability is

$$p_r = \left(1 - \frac{1}{n}\right)^{r-1} \qquad \text{for } r \geq 2,$$

   and we see that the formula is trivial for $r = 1$, hence it is valid for all $r \in \mathbb{N}$.

2) The first person can choose between n persons, the second one between $n-1$ persons, etc.. Person number j has $n+1-j$ possible successful choices. Hence, if $r < n$, the probability is

$$q_r = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n+1-r}{n} = \frac{(n-1)!}{(n-r)!} \cdot \frac{1}{n^{r-1}}.$$

Example 5.4 *Compute (1) and (2) of* Example 5.3*, by assuming that a rumour-monger never chooses to tell the rumour to the same person who just told it to himself.*

1) Just like in Example 5.3 the probability is 1 for $r = 1$.

   When $r > 1$, the following persons have $n-2$ possible successes, each of probability $\dfrac{1}{n-1}$, because we have $n-1$ choices, of which must be avoided. Thus we get the probability

$$p_r = \left(1 - \frac{1}{n-1}\right)^{r-1} \qquad \text{for } r > 1,$$

   and hence for all $r \in \mathbb{N}$. We assume implicitly that $n > 1$.

2) The first person can choose among n persons.
   The second person can choose among $n-1$ persons, because we have to exclude the person (corresponding to -1) who just told the rumour. Thus

$$q_1 = 1 \qquad \text{and} \qquad q_2 = 1.$$

   The j-th person, $j \geq 3$, has $n-j+1$ successes out of the $n-1$ possibilities, so when $r \geq 3$, then

$$q_r = \frac{n-2}{n-1} \cdot \frac{n-3}{n-1} \cdots \frac{n-r+1}{n-1} = \frac{(n-2)!}{(n-r)!} \cdot \frac{1}{(n-1)^{r-2}}.$$

Example 5.5 *In a classroom there are r pupils (none of them twins). Find the probability that all pupils have different birthdays (we assume that all 365 days have the same probability), and find approximative values of p_r for reasonable small values of r.*

The probability is

$$p_r = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{366-r}{365}.$$

Approximative values for $r = 1, \dots, 30$, are

$$\begin{array}{lll}
p_1 = 1,.000, & p_{11} = 0.859, & p_{21} = 0.556, \\
p_2 = 0.997, & p_{12} = 0.833, & p_{22} = 0.524, \\
p_3 = 0.992, & p_{13} = 0.806, & p_{23} = 0.493, \\
p_4 = 0.984, & p_{14} = 0.777, & p_{24} = 0.462, \\
p_5 = 0.973, & p_{15} = 0.747, & p_{25} = 0.431, \\
p_6 = 0.960, & p_{16} = 0.716, & p_{26} = 0.402, \\
p_7 = 0.944, & p_{17} = 0.685, & p_{27} = 0.373, \\
p_8 = 0.026, & p_{18} = 0.653, & p_{28} = 0.346, \\
p_9 = 0.905, & p_{19} = 0.621, & p_{29} = 0.319, \\
p_{10} = 0.883, & p_{20} = 0.589, & p_{30} = 0.294.
\end{array}$$

Remark 5.2 At the first glance it may be surprisingly that when there are 23 pupils. then we have the probability $1 - 0.493 = 0.507$ that at least two pupils have the same birthday. $\Diamond$

Example 5.6 *Assume that there are 100 lots in a lottery and only one prize.*
What is the probability of winning the prize, when one buys two lots?
If one instead buys one lot in each of two such lotteries, what is the probability of winning at least one of the two prizes?
Give an intuitive explanation of the fact that the two probabilities are not equal.

1) **1st variant.**

$$p = \frac{\text{number of bought lots}}{\text{total number of lots}} = \frac{2}{100} = 0.0200.$$

2nd variant.

$$\begin{aligned} p &= P\{\text{prize on the 1st lot}\} + P\{\text{loss on the 1st lot and prize on the 2nd lot}\} \\ &= \frac{1}{100} + \frac{99}{100} \cdot \frac{1}{99} = \frac{1}{100} + \frac{1}{100} = \frac{2}{100} = 0.0200. \end{aligned}$$

2) **1st variant.** The complementary event:

$$P\{\text{at least one prize}\} = 1 - P\{\text{no prize in 2 lotteries}\}$$
$$= 1 - \left(\frac{99}{100}\right)^2 = 1 - 0.9801 = 0.0199.$$

2nd variant. Direct computation, in which we identify the various possibilities:

$$\begin{aligned} &P\{\text{at least one prize}\} \\ &\quad = P\{\text{prize in the 1st lottery, loss in the 2nd lottery}\} \\ &\quad\quad + P\{\text{loss in the 1st lottery, prize in the 2nd lottery}\} \\ &\quad\quad + P\{\text{prizes in both lotteries}\} \\ &\quad = \frac{1}{100} \cdot \frac{99}{100} + \frac{99}{100} \cdot \frac{1}{100} + \frac{1}{100} \cdot \frac{1}{100} = 0.0199. \end{aligned}$$

When we apply the information that the average is the same when we compare the "average prizes", it follows that is twice as good to obtain 2 prizes (of probability 0.0001) then just 1 prize, the average can only remain the same if we also lower the probability in (2), hence the probability of (2) is smaller than the probability of (1).

Example 5.7 *There are two roads from X-town to Y-town, and two roads from Y-town to Z-town. In order to get from X-town to Z-town one must necessarily pass through Y-town. One day there is a blizzard over the area, and we assume that each of the four roads are closed with the probability p. Furthermore, we assume that the four roads are closed independently of each other.*
Find the probability that there is an open connection from X-town to Z-town.
A is going to drive from X-town to Z-town (without returning)- What is the probability that A is successful without any problems?

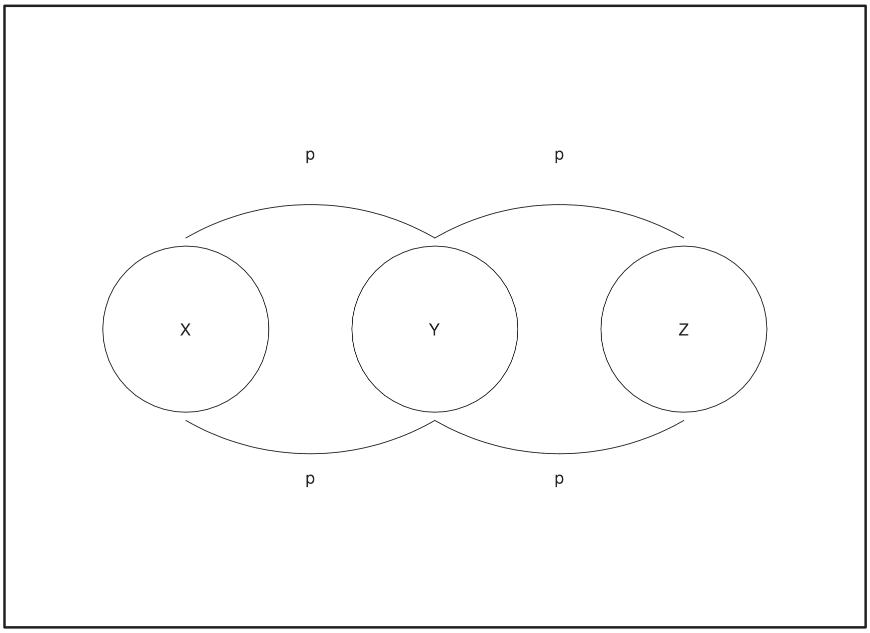


Figure 3: Graphical description of the roads between the cities.

The probability of the event that both roads from X to Y are closed is p^2.

The complementary event: The probability that one can at least go by one of the roads is then $1-p^2$.

Analogously for the roads from Y to Z.

Then

$$P\{\text{open road from X to Z}\} = P\{\text{open road from X to Y}\} \cdot P\{\text{open road from Y to Z}\} = \left(1-p^2\right)^2.$$

A can choose between the two roads from X to Y, each of probability $\frac{1}{2}$, hence

$$\begin{aligned} P\{\text{A gets from X to Y}\} &= \frac{1}{2}\,P\{\text{first road is open}\} + \frac{1}{2}\,P\{\text{second road is open}\} \\ &= \frac{1}{2}\,(1-p) + \frac{1}{2}\,(1-p) = 1-p, \end{aligned}$$

and analogously

$$P\{\text{A gets from Y to Z without problems}\} = 1 - p.$$

Then

$$\begin{aligned}&P\{\text{A gets from X via Y to Z without problems}\}\\ &\qquad = P\{\text{A gets from X to Y}\} \cdot P\{\text{A gets from Y to Z}\} = (1-p)^2.\end{aligned}$$

Obviously, when $0 < p < 1$, then $(1-p)^2 < \left(1-p^2\right)^2$.

Example 5.8 *Given a roulette, where the outcome of a single game is either red of probability p, or black of probability $q = 1 - p$. We assume that the games at the roulette are independent.*
We denote by X the stochastic variable which indicates the number of games in the first unchanged sequence of games. Thus, $X = 3$, if the sequence either is given by RRRB or BBBR.

1) Find for each $n \in \mathbb{N}$ the probability of $X = n$.

2) Find the mean of the stochastic variable X.

3) Find the minimum of the mean as a function of p.
Is this result reasonable?

1) Since $\{X = n\}$ means that we either first have n reds and then one black, or first n blacks and then one red, we get

$$P\{X = n\} = p^n q + q^n p = pq\left(p^{n-1} + q^{n-1}\right), \qquad n \in \mathbb{N}.$$

2) The mean value is computed in the usual way:

$$\begin{aligned} E\{X\} &= \sum_{n=1}^{\infty} n\,P\{X=n\} = pq \sum_{n=1}^{\infty}\left(n\,p^{n-1} + n\,q^{n-1}\right)\\ &= pq\left(\frac{1}{q^2} + \frac{1}{p^2}\right) = \frac{p}{q} + \frac{q}{p}.\end{aligned}$$

3) If we put $x = \dfrac{p}{q} > 0$, it follows that we shall minimize

$$E\{X\} = x + \frac{1}{x}, \qquad x > 0.$$

The minimum is obtained for $x = \dfrac{p}{q} = 1$, corresponding to $p = \dfrac{1}{2}$.

The result is reasonable because we for $p = \dfrac{1}{2}$ "often changes" between red and black. If e.g. $p > \dfrac{1}{2}$, then we shall get more reds in a row.

6 Binomial distribution

Example 6.1 *A coin is thrown $2n$ times, where we assume that each of the 2^{2n} possibilities have the same probability 2^{-2n}.*

1) *Find the probability that we obtain heads precisely n times.*

2) *Find the probability that we obtain heads precisely 2 times.*

3) *Find the probability of obtaining heads at least 2 times.*

1) We apply the binomial distribution with $p = q = \dfrac{1}{2}$, so

$$P\{X = n\} = \binom{2n}{n} \cdot \frac{1}{2^{2n}}.$$

The n successes can be chosen in $\binom{2n}{n}$ ways, each of probability 2^{-2n}.

2) Analogously,

$$P\{X = 2\} = \binom{2n}{2} \cdot \frac{1}{2^{2n}} = \frac{2n(2n-1)}{1 \cdot 2} \cdot \frac{1}{2^{2n}} = \frac{n(2n-1)}{2^{2n}}.$$

3) Considering the complementary event we get

$$\begin{aligned} P\{X \geq 2\} &= 1 - P\{X=0\} - P\{X=1\} = 1 - \binom{2n}{0} \cdot \frac{1}{2^{2n}} - \binom{2n}{1} \cdot \frac{1}{2^{2n}} \\ &= 1-(1+2n)\,\frac{1}{2^{2n}} = 1 - \frac{2n+1}{2^{2n}}. \end{aligned}$$

Example 6.2 *Some airplane companies allow for that some of their passengers who have booked a ticket do not show up at take off. Therefore, the airplane companies possibly use to overbook.*
We assume in the following that each passenger has the probability of $\frac{1}{10}$ *for not showing up and that this happens independently of the other passengers.*
Company A always sells 10 tickets to its small airplane of 9 seats.
Company B always sells 20 tickets to its larger airplane of 18 seats.
Find the probability that company A at take off must reject one passenger.
Find the probability that company B at take off must reject 1 or 2 passengers.
Which one is the larger probability?

Company A. We apply the binomial distribution, so

$$P\{k \text{ passengers appear}\} = \binom{10}{k} \left\{\frac{9}{10}\right\}^k \cdot \left\{\frac{1}{10}\right\}^{10-k}, \qquad k = 0, 1, \dots, 10,$$

thus k successes and $10-k$ failures, and the probability of success is $\frac{9}{10}$. Hence

$$P\{\text{A must reject}\} = P\{10 \text{ passengers appear}\} = \left\{\frac{9}{10}\right\}^{10} = 0.3487.$$

Company B. We again apply the binomial distribution with

$$P\{k \text{ passengers appear}\} = \binom{20}{k} \left\{\frac{9}{10}\right\}^k \cdot \left\{\frac{1}{10}\right\}^{20-k}, \qquad k = 0, 1, \dots, 20,$$

hence k successes, $20-k$ failures and the probability of success, $\frac{18}{20} = \frac{9}{10}$. Therefore,

$$\begin{aligned} P\{\text{B rejects}\} &= P\{20 \text{ passengers show up}\} + P\{19 \text{ passengers show up}\} \\ &= \left(\frac{9}{10}\right)^{20} + 20 \cdot \left(\frac{9}{10}\right)^{19} \cdot \frac{1}{10} = \left(\frac{9}{10}\right)^{19} \left\{\frac{9}{10} + \frac{20}{10}\right\} = \left(\frac{9}{10}\right)^{19} \cdot \frac{29}{10} = 0.3917. \end{aligned}$$

Remark. If one company sells 100 tickets with only seats for 90, the probability of rejection is

0.4513.

If it sells 400 tickets with seats for 360, the probability of rejection is

0.4756.

We see that the larger the airplane, the higher the probability of rejection. It can be proved that if the company sells $10n$ tickets with only space for $9n$, then the probability of rejection will increasingly go towards $\frac{1}{2}$ for $n \to +\infty$.

Example 6.3 *A person A fill in a row at random on a pools coupon, which means that for each of the 13 football games the probability is* $\frac{1}{3}$ *for success.*
Find the probability for A obtaining 13 successes, and the probability for A obtaining precisely 12 successes.
Another person B who knows more about football judges that he in each game has the probability $\frac{1}{2}$ *to guess right.*
Find the probability that B obtains 13 successes, and the probability that B gets precisely successes.
The expert in football C judges that she in each game has the probability of $\frac{2}{3}$ *for success.*
Find the probability that C obtains 13 successes, and the probability that C gets precisely 12 successes.

This is an exercise in the binomial distribution, where $N = 13$.

1) The probability of success is $p = \frac{1}{3}$, so

$$\begin{aligned} P_A\{13 \text{ successes}\} &= \frac{1}{3^{13}} = 0.000\,000\,627, \\ P_A\{12 \text{ successes}\} &= \binom{13}{12} \cdot \left\{\frac{1}{3}\right\}^{12} \cdot \frac{2}{3} = \frac{26}{3^{13}} = 0.000\,016\,3. \end{aligned}$$

2) The probability of success is $p = \frac{1}{2}$, so

$$\begin{aligned} P_A\{13 \text{ successes}\} &= \frac{1}{2^{13}} = 0.000\,244, \\ P_A\{12 \text{ successes}\} &= \binom{13}{12} \cdot \frac{1}{2^{13}} = \frac{13}{2^{13}} = 0.003\,17. \end{aligned}$$

3) The probability of success is $p = \frac{2}{3}$, so

$$\begin{aligned} P_A\{13 \text{ successes}\} &= \left(\frac{2}{3}\right)^{13} = 0.005\,14, \\ P_A\{12 \text{ successes}\} &= \binom{13}{12} \cdot \left\{\frac{2}{3}\right\}^{12} \cdot \frac{1}{3} = \frac{13}{2}\left(\frac{2}{3}\right)^{13} = 0.033\,4. \end{aligned}$$

7 Lotto

Example 7.1 *In lotto, 7 numbers are drawn among 1, 2, ... , 36. A young hopeful, who wants to travel around the world, chooses 7 numbers on a lotto coupon. Find the probability that he gets all 7 winning numbers, and find the probability that he gets 6 winning numbers.*

There are in total

$$\binom{36}{7} = 8\,347\,680$$

possible different lotto coupons.

There is only 1 winning coupon, so

$$p_7 = \frac{1}{\binom{36}{7}} \approx 0.000\,000\,12.$$

The expected number of weeks, before this event occurs is

$$\binom{36}{7} = 8\,347\,680 \text{ weeks } \approx 160\,532 \text{ years.}$$

Then we compute the number of possibilities of 6 winning numbers:

1) The six winning numbers can be chosen in $\binom{7}{6} = 7$ ways.

2) The wrong number must be chosen among the remaining $36 - 7 = 29$ numbers. This can be done in $\binom{29}{1} = 29$ ways.

In summary,

$$p_6 = \frac{\binom{7}{6}\binom{29}{1}}{\binom{36}{7}} = 0.000\,02432.$$

In general,

$$p_k = \frac{\binom{7}{k}\binom{29}{7-k}}{\binom{36}{7}}, \qquad k = 0, 1, \dots , 7,$$

thus

$$\begin{array}{llll} p_0 = 0.1870 & p_1 = 0.3983, & p_2 = 0.2987, & p_3 = 0.0996, \\ p_4 = 0.0153, & p_5 = 0.0010, & p_6 = 0.000\,024\,32, & p_7 = 0.000\,000\,12. \end{array}$$

We note that there are coupons with only 1 or 2 winning numbers in almost 70 % of all cases.

8 Huyghens' exercise

Example 8.1 *Two gamblers A and B throw by turns two dices. The game ends, when either A throws in total 6 (in which case A wins), or B throws in total 7 (in which case B wins B). If A is the first to throw, one shall find the probability that A wins, and also the probability that B wins.*
This exercise occurs in one of the very first books on the calculus of probability, written in 1656 by Huyghens.

	1	2	3	4	5	6
1	2	3	4	5	6^*	$7^\diamond$
2	3	4	5	6^*	$7^\diamond$	8
3	4	5	6^*	$7^\diamond$	8	9
4	5	6^*	$7^\diamond$	8	9	10
5	6^*	$7^\diamond$	8	9	10	11
6	$7^\diamond$	8	9	10	11	12

It follows from the table that we have in total $6 \cdot 6 = 36$ possibilities. The table also shows that 5 of these give the sum 6, and that 6 of them give the sum 7. This means that

$$\begin{aligned}
&P\{\text{A wins in a throw}\} = p_A = \frac{5}{36},\\
&P\{\text{A does not win in a throw}\} = q_A = \frac{31}{36},\\
&P\{\text{B wins in a throw}\} = p_B = \frac{1}{6},\\
&P\{\text{B does not win in a throw}\} = q_B = \frac{5}{6}.
\end{aligned}$$

The gamblers now throw by turns with A first. Then

$P\{\text{A wins in the 1st throw}\} = p_A,$
$P\{\text{B wins in the 2nd throw}\} = P\{\text{A does not win in the 1st throw}\}P\{\text{B wins in a throw}\} = q_A \cdot p_B,$
$P\{\text{A wins in the 3rd throw}\} = (q_A q_B)\, p_A,$
$P\{\text{B wins in the 4th throw}\} = (q_A q_B)\, q_A \cdot p_B,$

and then by induction,

$$P\{\text{A wins in throw number } 2n+1\} = (q_A q_B)^n\, p_A, \qquad n \in \mathbb{N}_0$$

$$P\{\text{B wins in throw number } 2n+2\} = (q_A q_B)^n\, q_A \cdot p_B, \qquad n \in \mathbb{N}_0.$$

Summing up we get

$$\begin{aligned} P\{\text{A wins}\} &= \sum_{n=0}^{+\infty} P\{\text{A wins in throw number } 2n+1\} = \sum_{n=0}^{+\infty} (q_A\, q_B)^n\, p_A = \frac{p_A}{1 - q_A\, q_B} \\ &= \frac{\frac{5}{36}}{1 - \frac{31}{36}\cdot\frac{5}{6}} = \frac{30}{216-155} = \frac{30}{61}, \end{aligned}$$

and

$$\begin{aligned} P\{\text{B wins}\} &= \sum_{n=0}^{+\infty} P\{\text{B wins in throw number } 2n+2\} = \sum_{n=0}^{+\infty} (q_A\, q_B)^n\, q_A\, p_B = \frac{q_A \cdot p_B}{1 - q_A\, q_B} \\ &= \frac{\frac{31}{36}\cdot\dfrac{1}{6}}{1 - \frac{31}{36}\cdot\frac{5}{6}} = \frac{31}{216-155} = \frac{31}{61}, \end{aligned}$$

As a check we see that the sum of the probabilities is

$$\frac{30}{61} + \frac{31}{61} = 1.$$

Also note that even if A has the advantage of being first, his smaller probability

$$p_A = \frac{5}{36} < p_B = \frac{1}{6}$$

nevertheless causes that B has a slightly larger probability of winning.

9 Balls in boxes

Example 9.1 *A box A contains 3 white balls and 1 black ball, and another box B contains 4 white balls. To the time 1 we draw at random 1 ball from each of the two boxes and interchange them. This interchanging is repeated at the times 2, 3, 4, ...*
Let p_n denote the probability that the black ball is in A after n such interchangings.

1) *Prove that* $p_n = \frac{1}{4} + \frac{1}{2}\,p_{n-1}$, $n = 2, 3, \dots$.

2) *Prove that* $p_n = \frac{2^n+1}{2^{n+1}}$, $n \in \mathbb{N}$.

3) *Find* $\lim_{n\to\infty} p_n$.

1) When $n = 2, 3, \dots$, it follows from the rule of total probability that

$$\begin{aligned} p_n &= P\{\text{black in A at time } n\} \\ &= P\{\text{black in A} \mid \text{black at time } n-1\} \cdot P\{\text{black in A at time } n-1\} \\ &\qquad + P\{\text{black in A} \mid \text{black in B at time } n-1\} \cdot P\{\text{black in B at time } n-1\} \\ &= \frac{3}{4}\,p_{n-1} + \frac{1}{4}\,\{1 - p_{n-1}\} = \frac{1}{4} + \frac{1}{2}\,p_{n-1}. \end{aligned}$$

2) If $n = 1$, then

$$p_1 = \frac{\text{number of whites in A}}{\text{number of balls in A}} = \frac{3}{4} = \frac{2^1+1}{2^2}.$$

If $n = 2$, then

$$p_2 = \frac{1}{4} + \frac{1}{2}\,p_1 = \frac{1}{4} + \frac{2^1+1}{2\cdot 2^n} = \frac{2+2+1}{2^3} = \frac{2^2+1}{2^3}.$$

Hence, the formula is true for $n = 1$ and for $n = 2$.

Now, assume that the formula is true for some $n \in \mathbb{N}$. Then

$$p_{n+1} = \frac{1}{4} + \frac{1}{2}\,p_n = \frac{2^n}{2^{n+2}} + \frac{2^n+1}{2^{n+2}} = \frac{2^n+2^n+1}{2^{n+2}} = \frac{2^{n+1}+1}{2^{n+2}},$$

and the formula is also true for $n+1$, and the claim follows by induction.

3) Clearly,

$$p_n = \frac{2^n+1}{2^{n+1}} = \frac{1}{2} + \frac{1}{2^{n+1}} \to \frac{1}{2} \qquad \text{for } n \to \infty,$$

which one also intuitively would expect.

10 Conditional probabilities, Bayes's formula

Example 10.1 *A family has 3 children, of which it is given that at least one is a boy. Find the probability that all 3 children are boys.*
(We assume that boys and girls have the same probability of being born).

There are in total $2^3 = 8$ possibilities. In 7 of these there is at least 1 boy, and there is only one possibility that they are all three boys. Since the 7 events have the same probability, the searched probability is $\frac{1}{7}$.

Remark 10.1 One frequently here sees Bayes's formula applied, in which case one obtains the wrong result $\frac{1}{4}$. The error is that one unconsciously assumes that (e.g.) the oldest child is a boy. $\Diamond$

Example 10.2 *A (true) dice is thrown, until it for the first time shows a six. Find the conditional probability that the first six occurs in the 2nd throw, given that the first six occurs in a throw of* even *number.*

It is well-known that we have the probability $\frac{1}{6}$ for getting a six, and the probability $\frac{5}{6}$ for not getting a six.

If the first six occurs in throw number $2n$, then the first $2n-1$ throws cannot have given a six. Thus the probability is

$$P\{\text{first six in throw number } 2n\} = \left(\frac{5}{6}\right)^{2n-1} \cdot \frac{1}{6}.$$

In particular,

$$P\{\text{first six in throw number } 2\} = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}.$$

Then we compute

$$\begin{aligned} P\{\text{first six in an even throw}\} &= \sum_{n=1}^{\infty} P\{\text{first six in throw number } 2n\} = \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{2n-1} \cdot \frac{1}{6} \\ &= \frac{5}{6} \cdot \frac{1}{6} \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^{2(n-1)} = \frac{5}{36} \cdot \frac{1}{1-\frac{25}{36}} = \frac{5}{36} \cdot \frac{36}{11} \qquad \left(=\frac{5}{11}\right), \end{aligned}$$

hence

$$P\{\text{first six in throw number } 2 \mid \text{first six in an even throw}\} = \frac{11}{36}.$$

Example 10.3 *In a talon there are 6 red cards, 3 of them being aces, and 6 black cards, 2 of them being aces. We draw one card at random from the talon.*

1) What is the probability that the drawn card is an ace, when we notice that it is red?

2) What is the probability that it is a red card, when we observe that it is an ace?

Red cards:	ace	ace	ace	$\diamond$	$\diamond$	$\diamond$
Black cards:	ace*	ace*	$*$	$*$	$*$	$*$

1) There are 6 red cards, 3 of which are aces, so the probability is

$$\frac{3}{6} = \frac{1}{2}.$$

2) There are 5 aces, 3 of which are red cards, so the probability is

$$\frac{3}{5}.$$

Example 10.4 *Three boxes A, B and C, contain red and black balls. Box A contains 2 red and 3 black balls, box B contains 1 red and 4 black balls, and box C contains 3 red balls and 1 black ball. We choose randomly a box, and from this box we choose randomly one of the balls. Assume that the drawn ball is red. Find the probability that the ball comes from box A.*

Let $\diamond$ denote a red ball and $*$ a black one. Then we get the following diagram,

$$\begin{array}{ccc} \diamond\ \diamond\ *\ *\ * & \diamond\ *\ *\ *\ * & \diamond\ \diamond\ \diamond\ * \\ \text{A} & \text{B} & \text{C} \end{array}$$

It follows that

$$P(A) = P(B) = P(C) = \frac{1}{3},$$

and that

$$P\{\text{red} \mid A\} = \frac{2}{5}, \qquad P\{\text{red} \mid B\} = \frac{1}{5}, \qquad P\{\text{red} \mid C\} = \frac{3}{4}.$$

Then by Bayes's formula,

$$\begin{aligned} P\{A \mid \text{red}\} &= \frac{P(A) \cdot P\{\text{red} \mid A\}}{P(A) \cdot P\{\text{red} \mid A\} + P(B) \cdot P\{\text{red} \mid B\} + P(C) \cdot P\{\text{red} \mid C\}} \\ &= \frac{\frac{1}{3} \cdot \frac{2}{5}}{\frac{1}{3}\left(\frac{2}{5} + \frac{1}{5} + \frac{3}{4}\right)} = \frac{\frac{2}{5}}{\frac{3}{5} + \frac{3}{4}} = \frac{\frac{2}{5}}{\frac{27}{20}} = \frac{\frac{40}{5}}{27} = \frac{8}{27}. \end{aligned}$$

Example 10.5 *Two boxes A_1 and A_2 contain w_1 white and b_1 black balls, and w_2 white and b_2 black balls, resp.. We draw at random one ball from each one of the boxes, and then at random one of the two balls.*
Find the probability that this ball is white.

Let A_i denote the event that a ball comes from box i, and let A denote the event that the ball is white. Since we choose 1 ball from each box, we get

$$P(A_i) = \frac{1}{2}, \qquad i = 1, 2.$$

Then by a simple counting,

$$P(A \mid A_i) = \frac{w_i}{w_i + b_i}, \qquad i = 1, 2,$$

hence by the law of total probability,

$$P(A) = P(A_1) P(A \mid A_1) + P(A_2) P(A \mid A_2) = \frac{1}{2} \frac{w_1}{w_1 + b_1} + \frac{1}{2} \frac{w_2}{w_2 + b_2}.$$

Example 10.6 *An information channel can transmit 0s and 1s, though some errors may occur. One expect that a sent 0 is changed with the probability* $\frac{1}{5}$ *to a 1, and that a sent 1 is changed with the probability* $\frac{1}{6}$ *to a 0. It is also given that in mean* $\frac{2}{3}$ *of all signals are 0s.*

1) Assuming that we receive a 0, what is the probability that a 0 was sent?

2) Assuming that we receive a 1, what is the probability that a 1 was sent?

We first write down Bayes's formula,

$$P\left(A_k \mid A\right) = \frac{P\left(A_k\right) \cdot P\left(A \mid A_k\right)}{\sum_i P\left(A_i\right) P\left(A \mid A_i\right)},$$

i.e. A_k and A "are interchanged".

1) Then we identify,

$$A_i = \{i \text{ sent}\}, \quad i = 0, 1, \qquad A = \{0 \text{ received}\}.$$

When this is put into Bayes's formula, we get

$$\begin{aligned} P\{0 \text{ sent.} \mid 0 \text{ received}\} &= \frac{P\{0 \text{ sent}\} \cdot P\{0 \text{ received} \mid 0 \text{ sent}\}}{P\{0 \text{ sent}\}P\{0 \text{ received.} \mid 0 \text{ sent}\} + P\{1 \text{ sent}\}P\{0 \text{ received} \mid 1 \text{ sent}\}} \\ &= \frac{\frac{2}{3} \cdot \frac{4}{5}}{\dfrac{2}{3} \cdot \frac{4}{5} + \frac{1}{3} \cdot \frac{1}{6}} = \frac{\frac{8}{15}}{\frac{8}{15} + \dfrac{1}{18}} = \frac{\frac{48}{90}}{\frac{53}{90}} = \frac{48}{53} \qquad (\approx 91\%). \end{aligned}$$

2) We here get by identification,

$$A_i = \{i \text{ sent}\}, \quad i = 0, 1, \qquad A = \{1 \text{ received}\}.$$

Then by insertion,

$$\begin{aligned} P\{1 \text{ sent} \mid 1 \text{ received}\} &= \frac{P\{1 \text{ sent}\} \cdot P\{1 \text{ received} \mid 1 \text{ sent}\}}{P\{1 \text{ sent}\}P\{1 \text{ received} \mid 1 \text{ sent}\} + P\{0 \text{ sent}\}P\{1 \text{ received} \mid 0 \text{ sent}\}} \\ &= \frac{\frac{1}{3} \cdot \frac{5}{6}}{\dfrac{1}{3} \cdot \frac{5}{6} + \frac{2}{3} \cdot \frac{1}{5}} = \frac{\frac{5}{18}}{\frac{5}{18} + \dfrac{2}{15}} = \frac{\frac{25}{90}}{\frac{37}{90}} = \frac{25}{37} \qquad (\approx 68\%). \end{aligned}$$

Example 10.7 *A factory buys 1000 light bulbs of type A, and 500 bulbs of type B, which are somewhat more expensive.*
For a randomly chosen bulb of type A there is the probability 0.6 of that it lasts longer than 2 months.
For a randomly chosen bulb of type B we have the probability 0.9 that it lasts longer than 2 months.
By mistake all bulbs are mixed together.
A bulb is chosen at random from the 1500 bulbs. Find the probability that this bulb will last for longer than 2 months.
If a bulb lasts for more than 2 months, what is the probability that it is of type A?

It follows directly that

$$
\begin{aligned}
&P\{\text{the bulb lasts in more than 2 months}\}\\
&\qquad = \frac{1000}{1500}\cdot 0.6+\frac{500}{1500}\cdot 0.9=\frac{2}{3}\cdot 0.6+\frac{1}{3}\cdot 0.9=0.7,
\end{aligned}
$$

and

$$
\begin{aligned}
&P\{\text{the bulb comes from A} \mid \text{the bulb lasts more than 2 months}\}\\
&= \frac{P\{\text{the bulb comes from A}\}\cdot P\{\text{the bulb lasts more than 2 months} \mid A\}}{P\{\text{the bulb lasts in more than 2 months}\}}=\frac{\frac{2}{3}\cdot 0.6}{0.7}=\frac{2\cdot 6}{3\cdot 7}=\frac{4}{7}\\
&\qquad (\approx 0.57).
\end{aligned}
$$

Example 10.8 *Given n slips of paper, each with one of the numbers 1, 2, ... , n. The n slips are randomly drawn one by one. Let k denote one of the numbers 2, 3, ... , $n-1$. Given that the number on the slip of number k is bigger than all the numbers of the previously drawn slips, what is the probability that the number is the largest one (i.e. is n)?*

Let a_k denote the number of the k-th drawn slip. Then

$$
\begin{aligned}
&P\{a_k=n \mid a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}\\
&\qquad = \frac{P\{a_k=n,\, a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}}{P\{a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}}=\frac{P\{a_k=n\}}{P\{a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}}.
\end{aligned}
$$

Since we have the same probability for that the largest number occurs in the 1st draw as in the 2nd draw, ... , in the k-th draw, we must have

$$P\{a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}=\frac{1}{k}.$$

Since

$$P\{a_k=n\}=\frac{1}{n},$$

we finally get that

$$P\{a_k=n \mid a_k>a_1,\, a_k>a_2,\, \dots,\, a_k>a_{k-1}\}=\frac{1/n}{1/k}=\frac{k}{n}.$$

Example 10.9 *One expects empirically in a collection of 1000 light bulbs the probability* $\frac{1}{6}$ *that there are 0, 1, 2, 3, 4 or 5 defective bulbs.*
We choose from a given collection of 1000 bulbs a random test consisting of 100 bulbs. Find an approximate estimate of the probability that there are no defective bulbs in this sample.
If we know that there are no defective bulbs in this random test, what is then the probability that there are no defective bulbs among the 1000 bulbs?

Let A_k denote the event that there are k defective bulbs among the 1000, where $k = 0, 1, 2, 3, 4, 5$. According to the assumption,

$$P(A_k) = \frac{1}{6}.$$

Let A be the event that there are no defective bulbs among the 100 bulbs. Then by the law of total probability,

$$(1)\quad P(A) = \sum_{k=0}^{5} P(A_k)\, P(A \mid A_k) = \frac{1}{6} \sum_{k=0}^{5} P(A \mid A_k)\,.$$

By a computation,

$$\begin{aligned}
P(A \mid A_0) &= 1, \\
P(A \mid A_1) &= \frac{999}{1000} \cdot \frac{998}{999} \cdots \frac{901}{902} \cdot \frac{900}{901} = \frac{9}{10} = 0.9, \\
P(A \mid A_2) &= \frac{998}{1000} \cdot \frac{997}{999} \cdots \frac{900}{902} \cdot \frac{899}{901} = \frac{9}{10} \cdot \frac{899}{999} = \frac{899}{999} \cdot P(A \mid A_1) = 0.8099, \\
P(A \mid A_3) &= \frac{898}{998} \cdot P(A \mid A_2) = 0.7288, \\
P(A \mid A_4) &= \frac{897}{997} \cdot P(A \mid A_3) = 0.6557, \\
P(A \mid A_5) &= \frac{896}{996} \cdot P(A \mid A_4) = 0.5898,
\end{aligned}$$

which by insertion into (1) gives

$$P(A) = 0.7807.$$

The question is answered by means of Bayes's formula,

$$P(A_0 \mid A) = \frac{P(A_0)\, P(A \mid A_0)}{\sum_{k=0}^{5} P(A_k)\, P(A \mid A_k)} = \frac{P(A_0)}{P(A)} = \frac{1}{6} \cdot \frac{1}{0.7807} = 0.2135.$$

11 Stochastic independency/dependency

Example 11.1 *A dice is independently thrown n times, where $n \geq 3$. Introduce for $i < j$ the events A_{ij} by*

$$A_{ij} = \{\text{same numbers in the } i\text{-th and the } j\text{-th throw}\}.$$

1) Compute $P(A_{ij})$.

2) Prove that the events A_{ij}, where $1 \leq i < j \leq n$, are pairwise independent, though not independent.

1) When the i-th throw has give the result $m \in \{1, 2, \dots, 6\}$, we have the probability $\frac{1}{6}$ of obtaining the same result m in the j-th throw, so

$$P(A_{ij}) = \frac{1}{6}.$$

2) Then consider the event $A_{ij} \cap A_{k\ell}$, where $i < j$ and $k < \ell$. Clearly, if all four numbers are mutually different, then clearly

$$P(A_{ij} \cap A_{k\ell}) = P(A_{ij}) \cdot P(A_{k\ell}).$$

If $(i, j) \neq (k, \ell)$, while i, resp. j, is equal to one of the numbers k, ℓ, the remaining two indices are still different, so given the number of the common index the probability is

$$P(A_{ij} \cap A_{k\ell}) = \frac{1}{6} \cdot \frac{1}{6} = P(A_{ij}) \cdot P(A_{k\ell}).$$

This proves that the events are pairwise independent.

Since $n \geq 3$, it suffices to consider the event $A_{12} \cap A_{13} \cap A_{23}$.
Given the result m in the 1st throw, we get the probability $\frac{1}{6}$ for obtaining m in the 2nd throw, and the probability $\frac{1}{6}$ for also to obtain m in the 3rd throw. Then the event A_{23} is automatically fulfilled, so

$$P(A_{12} \cap A_{13} \cap A_{23}) = P(A_{12} \cap A_{13}) = P(A_{12}) \cdot P(A_{13}) \neq P(A_{12}) \cdot P(A_{13}) \cdot P(A_{23}),$$

because $P(A_{23}) = \frac{1}{6} \neq 1$.
This shows that the events A_{ij} are not independent.

Example 11.2 *A coin is thrown n times, where $n \geq 2$. Each of the 2^n possible events are given the probability 2^{-n}. Let A denote the event that there in a series of n throws occur both heads and tails, and let B denote the event that we have at most one tail in the n throws. What can be said about the independency of A and B?*

It follows that

$$P(A) = 1 - P\{\text{all tails}\} - P\{\text{all heads}\} = 1 - \frac{1}{2^n} - \frac{1}{2^n} = 1 - \frac{1}{2^{n-1}} = \frac{2^{n-1}-1}{2^{n-1}},$$

and

$$P(B) = P\{\text{one tail}\} + P\{\text{all heads}\} = \frac{n}{2^n} + \frac{1}{2^n} = \frac{n+1}{2^n}.$$

The event $A \cap B$ describes that there is precisely one tail, hence

$$P(A \cap B) = P\{\text{one tail}\} = \frac{n}{2^n}.$$

This expression is equal to

$$P(A) \cdot P(B) = \frac{2^{n-1}-1}{2^n - 1} \cdot \frac{n+1}{2^n},$$

if and only if

$$n + 1 = 2^{n-1},$$

which happens if (and only if) $n = 3$. Hence we conclude that when $n = 3$, then A and B are independent, and that A and B are not independent, if $n \geq 2$ and $n \neq 3$.

12 Probabilities of events by set theory

Example 12.1 *Let A_1, A_2, ... , A_n denote n events. Prove by induction the formula*

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) &= \sum_i P(A_i) - \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \cdots \\ &\quad +(-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n). \end{aligned}$$

For $n=1$ we get the trivial identity $P(A_1) = P(A_1)$.

For $n=2$ we get

(2) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.

It follows from

$$\begin{aligned} P(A_1 \cup A_2) &= P(A_1 \setminus A_2) + P(A_2 \setminus A_1) + P(A_1 \cap A_2) \\ &= \{P(A_1 \setminus A_2) + P(A_1 \cap A_2)\} + \{P(A_2 \setminus A_1) + P(A_1 \cap A_2)\} - P(A_1 \cap A_2) \\ &= P(A_1) + P(A_2) - P(A_1 \cap A_2), \end{aligned}$$

that (2) holds.

Then assume that the formula holds for some $n \in \mathbb{N}$. (It follows from the above that it is true for $n=1$ and for $n=2$). Consider $n+1$ sets. Then it follows from (2) that

$$\begin{aligned} P\left(\bigcup_{i=1}^{n+1} A_i\right) &= P\left(\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right) \\ &= P\left(\bigcup_{i=1}^{n} A_i\right) + P(A_{n+1}) - P\left(\bigcup_{i=1}^{n} \{A_i \cap A_{n+1}\}\right) \\ &= \sum_{i\le n} P(A_i) - \sum_{i_1<i_2\le n} P(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3\le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \cdots \\ &= (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n) + P(A_{n+1}) \\ &\quad - \sum_{i\le n} P(A_i \cap A_{n+1}) + \sum_{i_1<i_2\le n} P(A_{i_1} \cap A_{i_2} \cap A_{n+1}) \\ &\quad - \sum_{i_1<i_2<i_3\le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{n+1}) \\ &\quad + \cdots + (-1)^n P(A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) \\ &= \sum_i P(A_i) - \sum_{i_1<i_2} P(A_i \cap A_{i_2}) + \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \cdots \\ &\quad +(-1)^n P(A_1 \cap A_2 \cap \cdots \cap A_{n+1}), \end{aligned}$$

which is the wanted formula with n replaced by $n+1$.

The formula then follows in general by induction.

Example 12.2 *Let $B_1, B_2, \dots, B_n$ denote n events. Prove the formula*

$$\begin{aligned} P\left(\bigcap_{i=1}^{n} B_i\right) &= \sum_i P(B_i) - \sum_{i_1<i_2} P(B_{i_1} \cup B_{i_2}) + \sum_{i_1<i_2<i_3} P(B_{i_1} \cup B_{i_2} \cup B_{i_3}) + \cdots \\ &\quad +(-1)^{n-1} P(B_1 \cup B_2 \cup \cdots \cup B_n). \end{aligned}$$

Hint: *The example can either be shown by induction, or one may use that*

$$\bigcap_{i\in J} B_i = \complement\left(\bigcup_{i\in J} \complement B_i\right)$$

for any index set, and then apply the result of Example 12.1.

The proof by induction follows the same pattern as the proof of Example 12.1, so we choose here the latter method, i.e. we assume that the result of Example 12.1 is known, Then

$$\begin{aligned} P\left(\bigcap_{i=1}^{n} B_i\right) &= P\left(\complement\left\{\bigcup_{i\in J} \complement B_i\right\}\right) = 1 - P\left(\bigcup_{i\in J} \complement B_i\right) \\ &= 1 - \left\{\sum_i P(\complement B_i) - \sum_{i_1<i_2} P(\complement B_{i_1} \cap \complement B_{i_2}) \right. \\ &\qquad \left. + \sum_{i_1<i_2<i_3} P(\complement B_{i_1} \cap \complement B_{i_2} \cap \complement B_{i_3}) + \cdots + (-1)^{n-1} P(\complement B_1 \cap \complement B_2 \cap \cdots \cap \complement B_n)\right\} \\ &= 1 - \left\{\sum_i P(\complement B_i) - \sum_{i_1<i_2} P(\complement(B_{i_1} \cup B_{i_2})) \right. \\ &\qquad \left. + \sum_{i_1<i_2<i_3} P(\complement(B_{i_1} \cup B_{i_2} \cup B_{i_3})) + \cdots + (-1)^{n-1} P(\complement(B_1 \cup B_2 \cup \cdots \cup B_n))\right\} \\ &= 1 - \left\{\sum_i \{1 - P(B_i)\} - \sum_{i_1<i_2} \{1 - P(B_{i_1} \cup B_{i_2})\} \right. \\ &\qquad \left. + \sum_{i_1<i_2<i_3} \{1 - P(B_{i_1} \cup B_{i_2} \cup B_{i_3})\} + \cdots + (-1)^{n-1}(1 - P(B_1 \cup B_2 \cup \cdots \cup B_n))\right\} \\ &= \sum_i P(B_i) - \sum_{i_1<i_2} P(B_{i_1} \cup B_{i_2}) \\ &\qquad + \sum_{i_1<i_2<i_3} P(B_{i_1} \cup B_{i_2} \cup B_{i_3}) + \cdots + (-1)^{n-1} P(B_1 \cup B_2 \cup \cdots \cup B_n) \\ &= \quad 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + 1. \end{aligned}$$

Then the formula follows from

$$1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + 1 = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j}(-1)^j = (1-1)^n = 0.$$

13 The rencontre problem and similar examples

Example 13.1 *The rencontre problem.*
In an airplane with n seats, n passengers – each provided with a numbered ticket – take at random their seats in the airplane. Find the probability p_n that no passenger is sitting on the right seat, and find an approximate value of this probability.

HINT: *Let A_k denote the event that the k-th passenger sits on the right seat. Then*

$$p_n = 1 - P\left(\bigcup_{k=1}^{n} A_k\right),$$

and we can apply the result of EXAMPLE 12.1.

This example can be found in many variants. Another formulation is the hat problem: *A sleepy cloakroom attendant distributes n hats between n gentlemen without bothering with whether the hats are in fact given to their owners; here p_n is the probability that no one gets his own hat.*
Its original formulation (which deals with card playing) goes back to the French mathematician Montmort (1708).

According to EXAMPLE 12.1,

$$\begin{aligned} p_n &= 1 - P\left(\bigcup_{k=1}^{n} A_k\right) \\ &= 1 - \sum_i P(A_i) + \sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) \\ &\qquad - \sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) + \cdots + (-1)^n P(A_1 \cap A_2 \cap \cdots \cap A_n). \end{aligned}$$

Here, $P(A_i) = \dfrac{1}{n}$, and

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j \mid A_i) = \frac{1}{n} \cdot \frac{1}{n-1} \qquad \text{for } i < j,$$

and in general,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \frac{1}{n} \cdot \frac{1}{n-1} \cdots \frac{1}{n+1-k}, \qquad \text{for } i_1 < i_2 < \cdots < i_k.$$

It follows that $\sum_i P(A_i) = 1$, and

$$\sum_{i_1<i_2} P(A_{i_1} \cap A_{i_2}) = \binom{n}{2} \frac{1}{n} \cdot 1n - 1 = \frac{1}{2!},$$

$$\sum_{i_1<i_2<i_3} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) = \binom{n}{3} \frac{1}{n(n-1)(n-2)} = \frac{1}{3!},$$

and in general

$$\sum_{i_1<i_2<\cdots<i_j} P\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}\right) = \binom{n}{j} \frac{1}{n(n-1)\cdots(n+1-j)} = \frac{1}{j!},$$

because we choose j seats out of n possibilities in $\binom{n}{j}$ ways.

We get by insertion,

$$p_n = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{(-1)^n}{n!} \to e^{-1} \approx 0.368 \qquad \text{for } n \to \infty.$$

For $n = 1$ it is obvious that the probability is $p_1 = 0$.
Then,

$$p_2 = \frac{1}{2}, \quad p_3 = \frac{1}{3}, \quad p_4 = \frac{3}{8} = 0.375, \quad p_5 = \frac{11}{30} \approx 0.367.$$

Since p_n is a section of the alternating series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

where the absolute value of the n-th term, $1/n!$, is obtained by removing the alternating factor $(-1)^n$, and which decreases towards zero, the error of the truncated series is always smaller than the first rejected term, i.e.

$$p_n = \frac{1}{e} + r_n, \qquad \text{where } |r_n| \le \frac{1}{(n+1)!}.$$

Example 13.2 *A box contains n slips of paper with the number 1 written on them, and m slips with the number 0, where $n \geq m-1$. The $n+m$ slips are drawn one by one without return. Find the probability that one never draws two slips with the number 0 following each other. Indicate in particular the probabilities in the case of $m=4$, $n=5$ and $m=4$, $n=8$.*

We have in total $m+n$ slips, corresponding to $m+n$ positions. We can choose n among these (the position of the 1s) in

$$\binom{m+n}{n} = \binom{m+n}{m} \qquad \text{måder.}$$

The successes are characterized by the fact that all zeros, with the exception of possibly the last one, are followed by a 1:

$$(0\,1),\ (0\,1),\ \dots,\ (0\,1),\ (0).$$

In this way we "fix" $m-1$ of the 1s of the total number of n. The remaining

$$n-(m-1) = n-m+1,$$

must be distributed among $n+1$ positions. (We have n of 1s and a free 0 to the right). This is equivalent to the placement of the m groups with a 0 on $n+1$ positions, which can be done in

$$\binom{m+n}{m} \qquad \text{ways.}$$

We conclude that the probability that we never successively draw two slips with the same number 0 is

$$\binom{n+1}{m} \Big/ \binom{m+n}{m}.$$

We get for $m=4$ and $n=5$, and for $m=4$ and $n=8$,

$$\binom{6}{4} \Big/ \binom{9}{4} = \frac{5}{42}, \qquad\qquad \binom{9}{4} \Big/ \binom{12}{4} = \frac{14}{55}.$$

Example 13.3 *In lotto, 7 winning numbers and 4 additional numbers are drawn among the numbers 1, 2, 3, ... , 36. It happens quite often that there amongst the 7 winning numbers are two or more neighbouring numbers, e.g. both 6 and 7 are winning numbers.*

1) Find the probability that there are no neighbouring numbers among the 4 additional numbers.

2) Find the probability that there are no neighbouring numbers among the 7 winning numbers.

3) Find the probability that there are no neighbouring numbers among the 11 drawn numbers.

HINT: *The example is related to the previous example.*

We assume in general that $n \leq 18$. Then we can draw n numbers among 36 in

$$\binom{36}{n} \qquad \text{ways.}$$

In the case where there are no neighbouring numbers in this sample, then we have two possibilities:

(a) The number 36 is not chosen, so the successor of each of the n numbers cannot be chosen. This gives n constraints, and we get $36 - n$ possibilities for the n places. We can choose n out of $36 - n$ in

$$\binom{36-n}{n} \quad \text{ways.}$$

(b) If 36 is chosen, then we are left with 35 places and $n - 1$ constraints, hence $35 - (n - 1) = 36 - n$ places for $n - 1$ pairs. We can choose $n - 1$ from $36 - n$ in

$$\binom{36-n}{n-1} \quad \text{ways.}$$

Summarizing we get

$$\binom{36-n}{n} + \binom{36-n}{n-1} = \binom{37-n}{n} \quad \text{successes,}$$

so the probability is

$$\binom{37-n}{n} \Big/ \binom{36}{n}.$$

1) If $n = 4$, then the probability is

$$\binom{33}{4} \Big/ \binom{36}{4} = \frac{248}{357} \approx 0.694.$$

2) If $n = 7$, then the probability is

$$\binom{30}{7} \Big/ \binom{36}{7} = \frac{16965}{69564} \approx 0.244.$$

3) If $n = 11$, then the probability is

$$\binom{37-n}{n} \Big/ \binom{36}{n} = \frac{2185}{169911} \approx 0.0129.$$

14 Strategy in games

Example 14.1 *Andrew has bought 3 unusual dices. Dice A has on its 6 surfaces the numbers 1, 6, 11, 12, 13, 14, dice B has analogously the numbers 2, 3, 4, 15, 16, 17, and dice C has the numbers 5, 7, 8, 9, 10, 18. All 3 dices are considered as true, i.e. be throwing any of the three dices each surface has the probability* $\frac{1}{6}$ *for showing up.*
Andrew suggest Peter to play a game with the following rules: They each choose one dice, throw it, and the one who gets the highest number is winning a prize. Peter is allowed first to choose among the 3 dices, and afterwards Andrew can choose between the two remaining dices. This game is repeated 12 times. Who has won most prizes?

The example reminds very much of the well-known game of "stone–paper–scissors", so a qualified guess is that Andrew will win most prizes, because he knows Peters strategy. We shall now prove this.

We get by a computation that

$$\begin{aligned}
P\{A>B\} &= (P\{A{=}14\}+P\{A{=}13\}+P\{A{=}12\}+P\{A{=}11\}+P\{A{=}6\})\times \\
&\qquad \times(P\{Y=4\}+P\{Y=3\}+P\{Y=2\}) \\
&= 5\cdot\frac{1}{6}\cdot 3\cdot\frac{1}{6}=\frac{5}{12}, \\
P\{B>C\} &= (P\{B=17\}+P\{B=16\}+P\{B=15\})\times \\
&\qquad \times(P\{C{=}10\}+P\{C{=}9\}+P\{C{=}8\}+P\{C{=}7\}+P\{C{=}5\}) \\
&= 3\cdot\frac{1}{6}\cdot 5\cdot\frac{1}{6}=\frac{5}{12}, \\
P\{C>A\} &= P\{C=18\}\cdot(P\{A=14\}+P\{A=13\}+P\{A=12\}+ \\
&\qquad\qquad +P\{A=11\}+P\{A=6\}+P\{A=1\}) \\
&\qquad +(P\{C=10\}+P\{C=9\}+P\{C=8\}+P\{C=7\})\times \\
&\qquad\qquad \times(P\{A=6\}+P\{A=1\}) \\
&\quad +P\{C=5\}\cdot P\{A=1\} \\
&= \frac{1}{6}\cdot 6\cdot\frac{1}{6}+4\cdot\frac{1}{6}\cdot 2\frac{1}{6}+\frac{1}{6}\cdot\frac{1}{6}=\frac{6+8+1}{36}=\frac{15}{36}=\frac{5}{12}.
\end{aligned}$$

Summing up we get

$$P\{A>B\}=\frac{5}{12},\qquad \text{i.e.}\qquad P\{B>A\}=\frac{7}{12},$$

$$P\{B>C\}=\frac{5}{12},\qquad \text{i.e.}\qquad P\{C>B\}=\frac{7}{12},$$

$$P\{C>A\}=\frac{5}{12},\qquad \text{i.e.}\qquad P\{A>C\}=\frac{7}{12}.$$

Hence, Andrew's strategy must be:

1) If Peter chooses A, then Andrew chooses B.

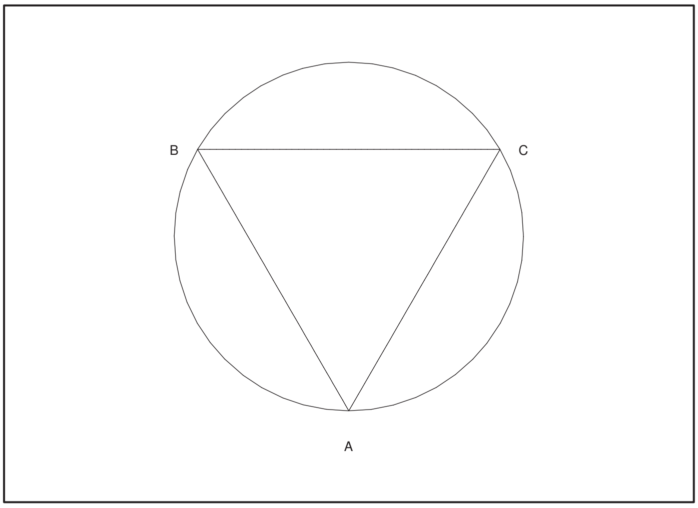
2) If Peter chooses B, then Andrew chooses C.

3) If Peter chooses C, then Andrew chooses A.

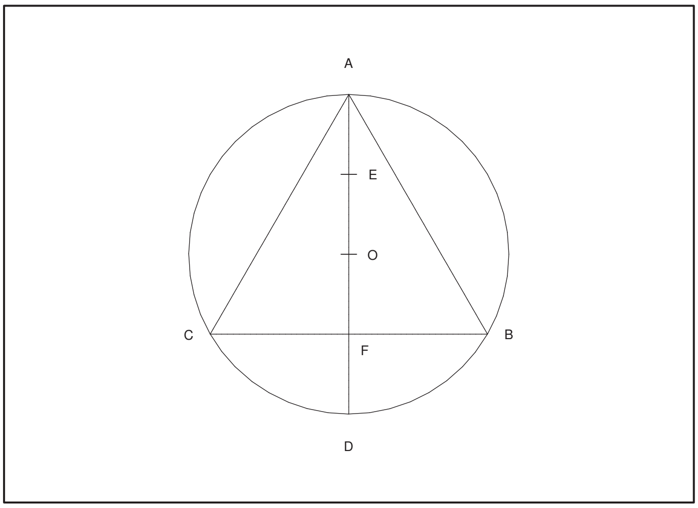
In all three cases Andrew has the probability $\frac{7}{12}$ of winning, so we may expect that he after 12 games has won 7 times.

15 Bertrand's paradox

Example 15.1 *A cord is drawn at random in a given circle. What is the probability p that the cord is longer than the side length of an inscribed equilateral triangle in the circle?*



Solution 1. *One end point of the cord is denoted by A, while the other one is chosen randomly on the circle. If it lies on the smallest of the arcs BC, then the cord is longer then the side length of the equilateral triangle, hence* $p = \dfrac{1}{3}$.



Solution 2. *The random cord is perpendicular so some diameter AD. The cord is longer than the side length of the equilateral triangle, if it cuts AD between E and F, hence* $p = \dfrac{1}{2}$.

Solution 3. *The midpoint of the random cord is chosen anywhere in the open of the disc. Then the* *cord is longer than the side length of the equilateral triangle, if the midpoint lies inside the inscribed circle of radius* $\frac{r}{2}$*, hence* $p = \frac{1}{4}$.

Explain this discrepancy.

The example is known as Bertrand's paradox.

In the first solution we use the arc length on the circle divided by $2\pi r$ as our probability measure.

In the second solution we use the length on AD, divided by the length of AD, as our probability measure.

In the third solution we use the area, divided by the area of the whole disc πr^2, as our probability measure.

These three probability measures are clearly mutually different, explaining why we get different results.

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